Higher rank Wilson Loops in the ABJM Matrix Model

Jonathan Cookmeyer
Haverford College
Advisor: Jim Liu
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1 Introduction

Since the first paper on the subject, the AdS/CFT correspondence has become one of the most important ideas of study in theoretical high energy physics. Introduced by Maldacena in 1998, it is a hypothesized equivalence between a string theory in Anti-de Sitter space (AdS) and a Conformal Field Theory (CFT) in one lower dimension [6]. One of the more well studied instantiations is the duality between $\mathcal{N} = 4$ supersymmetric Yang Mills CFT and string theory in $\mathrm{AdS}_5 \times S^5$.

More recently, the ABJM conformal field theory proposed by Aharony, Bergman, Jafferis, and Maldacena is conjectured to be dual to an M-theory in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ [1]. The partition function of this theory was written as an integral over matrices in [5] through the technique of localization. This matrix model partition function can then be used to calculate the expectation value of Wilson Loop observables—non-local observables containing information about the underlying field theory in which they are calculated.

To provide insight into M-theory as well as the ABJM/$\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ duality, the goal of my project was to calculate certain expectation values of Wilson Loop observables in ABJM to compare with calculations done in AdS. We were able to find expressions of the leading and sub-leading order behavior for the observables of interest. Our leading order calculations match results in [2].

2 Calculations of m-symmetric and m-antisymmetric Wilson Loops

To calculate any expectation value in ABJM, we need a partition function. Through the powerful technique of localization, the ABJM partition function can be written as [5]

$$Z(N) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} \frac{d\mu_i}{2\pi} \frac{d\nu_i}{2\pi} \frac{\prod_{i<j} \left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2 \left(2\sinh\frac{\nu_i-\nu_j}{2}\right)^2}{\prod_{i,j}\left(2\cosh\frac{\mu_i-\nu_j}{2}\right)^2} \exp\left[\frac{ik}{4\pi}\sum_i(\mu_i^2-\nu_i^2)\right]. \quad (1)$$

The observables we are interested in are the expectation values of the circular 1/6 m-symmetric and m-antisymmetric Wilson Loop values. In ABJM, a Wilson Loop observable is given by

$$W_{\mathcal{R}} = \frac{1}{\dim[\mathcal{R}]} \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \oint \left(iA_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}| M^I_J C_I \bar{C}^J \right) ds, \quad (2)$$

where A_μ is one of the gauge fields, and C_I are bifundamentals. Here $\mathcal{R}$ denotes a representation of $U(N)$. To find the m-symmetric or m-antisymmetric circular Wilson Loop, we pick the irreducible representation of rank m that is fully symmetric or antisymmetric. For

$M = \mathrm{diag}(1,1,-1,-1)$ and a circular path integral, 1/6 of the supersymmetries of ABJM are preserved [2]. Under localization, the operators become

$$W_{1/6}^{S_m} = \frac{1}{\dim[S_m]} \sum_{1 \le i_1 \le \ldots \le i_m \le N} \exp[\mu_{i_1} + \ldots + \mu_{i_m}], \tag{3}$$

$$W_{1/6}^{A_m} = \frac{1}{\dim[A_m]} \sum_{N \ge i_1 > \ldots > i_m \ge 1} \exp[\mu_{i_1} + \ldots + \mu_{i_m}] \tag{4}$$

where $\dim[S_m]$ and $\dim[A_m]$ are the dimensions of the representation. The main difference is the range of i_j. These observables are of interest because on the AdS side, they are simply the action of D2- and D6-branes respectively.

To calculate expectation values, we would insert (3) and (4) separately into the integral in (1), and normalize by 1/(1). Because of the difficulty of manipulating the expressions for $W_{1/6}^{S_m,A_m}$, to calculate the expectation values, we use ideas first introduced in [3] that rewrite the observables in terms of a contour integral of a generating function.

To do the calculations, there were two possible limits we could take involving the parameters N and k in (1). The type IIA limit brings $N \to \infty$ and $k \to \infty$ such that $N/k = \lambda$ is fixed. The M-theory limit takes $N \to \infty$ while fixing k, so $\lambda \to \infty$. It turns out that working in the IIA limit is difficult in full generality, so we work in the M-theory limit. We verified that the IIA limit calculation with large λ should produce the same results to leading order.

The main tool in these calculations is the saddle-point approximation or steepest decent approximation of an integral. Essentially, for a complex function $g(x)$ or a real function analytically continued to the complex plane, we can write

$$\int d^n x\, e^{Ng(\boldsymbol{x})} \approx \left(\frac{2\pi}{N}\right)^{n/2} \sqrt{\frac{1}{\det \mathbf{H}(\boldsymbol{x_0})}} e^{Ng(\boldsymbol{x}_0)} \qquad N \to \infty \tag{5}$$

where $Dg(\boldsymbol{x}_0) = \mathbf{0}$ or, equivalently, $\boldsymbol{x}_0$ is a saddle point of $g(\boldsymbol{x})$, and $\mathbf{H}$ is the Hessian matrix of $g(\boldsymbol{x})$.

Using the saddle point of (1) derived in the M-theory limit [4], we proceed with our calculations. Using the techniques in [3], we calculate $W_{1/6}^{S_m,A_m}$ where $m/N = f$ is fixed. That is, we are looking at $m \sim N \gg 1$. The antisymmetric representation proceeds in a quite straight forward manor, and we find

$$W_{1/6}^{A_m} \sim \sqrt{\frac{1}{N}} \exp\left[N\pi\sqrt{2\lambda} f(1-f) + \frac{N\pi}{12\sqrt{2\lambda}} + \frac{1}{4}\ln 2\lambda\right]. \tag{6}$$

We are able to do the calculation beyond the leading order, but only the leading order calculation has been successfully computed on the AdS side. The leading order behavior matches the calculation of the D6-brane in the type IIA limit [2]. This result also has $f \to 1 - f$ symmetry, which we expect.

The symmetric case is a bit more difficult because the standard saddle point procedure does not work. After several weeks of playing around with the integral and trying various things, we found a technique that worked. The trick was first to change the contour along which we were integrating, then to expand the integral not around the saddle point but around a nearby point. Having done that, after some manipulations, we were able to express the integral as an asymptotic series in λ, whose behavior we could control. In the end, the symmetric representation is given as

$$W_{1/6}^{S_m} \sim \exp\left[fN\pi\sqrt{2\lambda} + \frac{N\pi}{12\sqrt{2\lambda}} - \ln(Nf^2 2\pi\sqrt{2\lambda})\right]. \tag{7}$$

On the AdS side, the calculation of the D2-brane was attempted in [2], but only for certain values of m was a saddle point found.

It is interesting to note that the structure of $W_{1/6}^{S_m}$ and $W_{1/6}^{A_m}$ is quite similar. Recalling that $\lambda = N/k \sim N$, we see that there are three terms which grow as $N^{3/2}$, $N^{1/2}$, and $\ln N$. The term growing as $N^{1/2}$ is in fact the same for both cases.

It is also interesting that, to leading order in N and, in the antisymmetric case, for $m \ll N$, we see that $W_{1/6}^{S_m,A_m} \sim \exp[m\pi\sqrt{2\lambda}]$. This expression matches the m times would Wilson Loop, which is calculated in [7], for example. In the $\mathcal{N} = 4$ supersymmetric Yang Mills case, this limiting behavior was seen as well.

Our results serve as important predictions for M-theory calculations on the AdS side as well as interesting comparison for string theory calculations done in the Type IIA limit.

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