Gapless Edge Modes in Metallic Systems with Conformal Symmetry

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Quantum Hall Effect

- Phase of wavefunction responsible for many quantum effects
- One such example is insulators with protected surface states
- First example is quantum hall effect (QHE)
- Turn on B field perpendicular to current, induced voltage perpendicular to original current
- Bulk of system is insulating, $\sigma_{xx} = \sigma_{yy} = 0$
- Quantized hall conductance observed at low T, high B $\sigma_{xy} \neq 0$. Units of $\frac{e^2}{h}$
- Current on edge in direction determined by B field
- Used to determine $\alpha$ in verification of QED prediction without using QED!
Surface States

QHE

QSHE and 3D TI

Figure 1 from S. Zhang, Physics 1, 6 (2008)
Band Theory

- Bloch’s theorem: eigenfunctions of solid are periodic in the crystal lattice
- Energy band: set of energy eigenvalues corresponding to each reciprocal lattice vector in infinite limit
- Can define an effective mass or charge carrier related to curvature of band
- Kane Mele model of QSHE is example of band structure with nontrivial topology.
- Nontrivial topology → Edge states

Figure 1 from D. Hsieh et al., Science 323, 919 (2009). [Primary P.I.: M. Zahid Hasan (Princeton University)]
Surface States in Semimetals

- What happens in a band structure with linear dispersion near a point?
  
- Effective mass of band is zero
  
- Charge carriers are massless Dirac or Weyl fermions
  
- Obey the Dirac equation for massless particle \((i\gamma^\mu \partial_\mu - m)\Psi(x, t) = 0\)
  
- Dispersion is relativistic even though charge carriers move only \((1/300)c!\)
  
- Dirac point is critical point between a trivial and topological insulator
  
- Example: Graphene has a Dirac point in 2D
  
- Three dimensions: Weyl or Dirac semimetal (two Weyl points at the same point protected by symmetry)
  
- Systems with these points are all metallic, but still have protected edge modes
  
- Unlike for insulators location and presence not completely explained by topology
  
- Presence and location of edge modes determined by conformal symmetry
  
- Also distinct in that these edge states depend on boundary conditions
Conformal Symmetry

- Invariance under conformal transformations
- Transformations preserve angles, scale invariance
- Contains dilations and the special conformal transformation
- Critical point → system looks the same at all distances
- Conformal invariance be used to describe phase transition close to critical point
- Especially elegant in 2D systems
- Just Analytic functions defined on a cylinder (periodic functions)
  \[ f(z) = \sum_k a_k e^{kz} = \sum_k a_k e^{kx} e^{iky} \]
- Dirac point is approximately linear, conformal symmetry close to that point
Consider hopping between neighboring A sites (red) and B sites (blue)

- Hopping amplitude $t$
- Can have left and right moving zero modes on edge

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

- Edge mode occurs on side determined by boundary conditions
- Must have boundary conditions so $\psi_R$ and $\psi_L$ are holomorphic to preserve conformal symmetry on the boundary
- Commutation relation from Heisenberg equation of motion defines constraints

$$\frac{d}{dt} c_i(t) = \frac{i}{\hbar} [H, c_i(t)]$$

$$E = \pm 4t \sqrt{\frac{1}{2} \sin^2 \left(\frac{k_x}{2}\right) \sin^2 \left(\frac{k_y}{2}\right) + \frac{1}{2} \cos^2 \left(\frac{k_x}{2}\right) \cos^2 \left(\frac{k_y}{2}\right)}$$
Edge Modes and Eigenfunctions

- Can verify theoretical predictions using edge calculations
- Decay constant approximately equal to wave vector for small $k$
- Diagonal edge does not have an edge mode because of boundary conditions
Honeycomb Lattice

- Two different types of edges for honeycomb lattice
- Arm chair edge does not normally have an edge mode but does if we remove one of the sites from the boundary.

Figure Role of edges in the electronic and magnetic structures of nanographene Toshiaki Enoki 2012 Phys. Scr. 2012 014008 doi:10.1088/0031-8949/2012/T146/014008
Weyl Semimetal

- 3D Weyl Hamiltonian \( H = v_{ij} k_i \cdot \sigma_j \)
- Weyl points come in pairs of opposite chirality (right and left)
- Separated by symmetry breaking and connected by Fermi arc
- Fermi arc is topological, will not disappear unless Weyl points are brought to the same \( k \) point and annihilated
- If Weyl points occur at same \( k \) and are protected by symmetry, can have Dirac semimetal, a 3D analog of graphene
- Boundary conditions determine the angle at which the fermi arc occurs
- In the past, thought that if Weyl points of opposite chirality were projected to \( k_z \) which were directly opposite each other, there would be no edge modes

Figure 1 from L. Balents, Physics 4, 36 (2011).

Angle of Conformal Mode

- Angle of edge mode in the xy plane determined by boundary conditions
- Gives parameters (angles and phases) to do a unitary transformation of the Hamiltonian
- Can write a linear combination of original left and right moving modes explicitly as a holomorphic function
- Use model of a cubic lattice, from Jiang, J.H., Tunable topological weyl semimetal from simple cubic lattices with staggered fluxes arXiv:1112.5943v7(2011)
- Make a cut along yz plane and find an edge mode at $\varphi=\pi/2$
- Still working on cases with different boundary conditions
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- Semiconductor Physics Group, Department of Physics, University of Cambridge (Quantum Hall Effect)
Conformal Symmetry and the Complex Plane

- In two dimensions, all holomorphic functions have conformal symmetry.
- Can create conformal mappings between holomorphic functions that preserve angles.
- Holomorphic $\rightarrow$ differentiable everywhere in complex plane.
- Satisfies Cauchy Riemann Equations.
- Implies that the real and imaginary parts satisfy Laplace’s equation.
- Implies orthogonality.
- Conformal symmetry more restricted for $d>2$.
- Can write a complex variable and its conjugate $\bar{z} = x - iy$ $z = x + iy$.
- Can write a function as $f(z, \bar{z}) = u(x, y) + iv(x, y)$.
- Cauchy-Riemann relations imply holomorphic function can only be a function of $z$ or its complex conjugate, but not both.
Symmetry in Electronic Structure

- Landau Theory: Phase transitions \(\rightarrow\) symmetry breaking
- Example: liquid-solid transition, continuous translation \(\rightarrow\) discrete lattice
- Distinction between discrete and continuous symmetry
- Noether’s Theorem says that for every continuous symmetry (parameter does not appear in the Lagrangian, we get a conserved quantity in the equation of motion.
- In quantum mechanics, an operator that commutes with the Hamiltonian is a symmetry operator
- Symmetry can give estimates of parameters and degeneracies
- Continuous symmetry is only approximate
- Example: Coulomb potential makes s and p states degenerate, but not exactly when considering fine/hyperfine corrections

Topology

- Topological phases transitions: QHE, QSHE
- Topological property is something preserved under continuous transformations
- Systems with the same fundamental group are homotopic
- Topology exact, but insensitive to adiabatic changes (boundary conditions)
- A donut and a coffee cup are equivalent topologically

Tight-Binding Model

- Assume electrons are tightly bound to sites, hopping factor $t$ (real or complex) and phase factor from translations between sites.
- Can just include nearest neighbors, but next nearest neighbors may also be important (Haldane and Kane-Mele models).
  \[ H = -t \sum_{\langle i, j \rangle, \sigma} c_{i, \sigma}^\dagger c_{j, \sigma} + h.c. \]
- Can also include spin orbit terms (Kane-Mele model).
- Near point, can do what is called a $k \cdot p$ expansion to get the behavior very close to that point.
- For a Weyl point this expansion looks like the following written with Pauli matrices.
  \[ H = v_{ij} k_i \cdot \sigma_j \]
- A Dirac Hamiltonian is in essence two copies of a Weyl Hamiltonian and is written with the 4x4 $\Upsilon$ matrices which are direct products of Pauli matrices.
- This formalism uses second quantization, create and destroy particles using creation and annihilation operators like for the quantum harmonic oscillator.
Checkerboard Lattice Model

- Consider hopping between neighboring A sites (red) and B sites (blue)
- Complex hopping amplitude $t$
- Analogous to a magnetic field
- Zig-zag edge mode occurs on side determined by boundary
- Diagonal edge does not have an edge mode
- Tight-Binding Model Below

\[
H(k) = \sum_k \begin{pmatrix} c^\dagger_a & c^\dagger_b \end{pmatrix} \begin{pmatrix} 0 & -4t \left( \frac{\cos \left( \frac{k_x}{2} \right) \cos \left( \frac{k_y}{2} \right)}{\sqrt{2}} - i \frac{\sin \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right)}{\sqrt{2}} \right) \\ -4t \left( \frac{\cos \left( \frac{k_x}{2} \right) \cos \left( \frac{k_y}{2} \right)}{\sqrt{2}} + i \frac{\sin \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right)}{\sqrt{2}} \right) & 0 \end{pmatrix} \begin{pmatrix} c_a \\ c_b \end{pmatrix}
\]

\[
E = \pm 4t \sqrt{\frac{1}{2} \sin^2 \left( \frac{k_x}{2} \right) \sin^2 \left( \frac{k_y}{2} \right) + \frac{1}{2} \cos^2 \left( \frac{k_x}{2} \right) \cos^2 \left( \frac{k_y}{2} \right)}
\]