Analysis of Out-of-Plane Deformations in Maxwell Lattices

Robert Voinescu, Northern Arizona University
Maxwell Lattices

- The concept of mechanical stability was first truly studied by James Clerk Maxwell in 1864.
- Sites linked by rigid struts cannot be stable unless the number of constraints, represented by the struts, exceeds the number of degrees of freedom of the sites minus the number of rigid translations and rotation.

\[ N_b = z \frac{N}{2} \]

\[ \langle z \rangle > d \times N - \frac{d(d + 1)}{2} \]

\[ \langle z \rangle = 2d \]
Maxwell Lattices

\[ z = 3 < 2d \]
Floppy
Extensive # of floppy modes

\[ z = 2d \]
“Maxwell Lattices”
How many floppy modes?

\[ z = 6 > 2d \]
Over-constrained
No floppy modes
Applications

- Elastic networks capture the essential physics of many interesting systems

This Maxwellian argument and its variants are used extensively in engineering literature and have proven useful in the study of

- Rigidity Percolation
- Jamming Transitions in Granular Material
- Glass Transitions
Applications

❖ An interesting application related to my research is in the development of materials with a tunable bulk modulus. Which could be applied to something like reducing damage of impact in car crashes or re-entry.

❖ Note our analysis does not take a particular material into account. Thus all results here can be generalized to most materials at some level.

❖ The ultimate goal is to work backwards from a desired surface to the underlying lattice and necessary strains.

Emergent Complexity

- Though locally the interactions can be described simply it is when we look at larger structures and see how each vertex interacts with the other that we see some emergent complexity. It is even the case for very simple mechanical structures that we can get very interesting developments.
- The resultant floppy modes are non trivial.
- Finding a linkage that pre-describes a curve is difficult.
- Interesting problem for Physicists, Engineers and Algebraic Geometers.

Review of Continuum Mechanics

\[ u_{i k} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \]

\[ F_i = \frac{\sigma_{i k}}{\partial x_k} \]
First Order Analysis of Frameworks

\[ \mathbf{C} \cdot \mathbf{u} = e \quad \text{rank}(\mathbf{C}) + N_0 = Nd \]

\[ \mathbf{Q} \cdot t = f \quad \text{rank}(\mathbf{Q}) + N_s = N_c \]

\[ \mathbf{C} = \mathbf{Q}^T \]

\[ N_0 - N_s = Nd - N_c \]

\[ H = \frac{1}{2} \mathbf{u} \cdot \mathbf{D} \cdot \mathbf{u} \]

\[ e_{nm} = b_{nm} \cdot (\mathbf{u}_n - \mathbf{u}_m) \]

- Elements in the matrix \( \mathbf{C} \) are determined by the extension of the springs.
- \( \mathbf{Q} \) encodes the net force on the system given the tension in each bond.
- The kernel of the configuration matrix contains the differential states of self stress.
- The kernel of the compatibility matrix contains the differential zero modes.
- Code was written utilizing this derivation to search for and count the floppy modes.
Internal Mechanisms

- My work focused on kagome and checkerboard lattices and their variations.
- Exhibited to the left is each lattice’s mode of uniform deformation.
- These are auxetic lattices.
Computation

A major part of my work this summer was to create code that we could use to find the effects of certain boundary conditions.

Initial Problem and Runtimes

- Run times for desired lattices would take multiple hours even for the smallest cases.
- A lot of this time was not just spent in minimizing our solution, but defining the terms and simplifying the resulting equation to something usable itself was time intensive.

Minimization

- Now, the code takes a point and utilizing a lagrange polynomial (3 point central) finds the derivative about the point and utilizes that to minimize.

Cross terms

- To avoid the unnecessary terms my code given a lattice generates the incidence matrix and uses that to knock out the values that would just cancel in calculating the derivative.
Computation

- To ensure convergence the step size is dynamically actuated.
- The Code is optimized to run on 24 cores on the flux cluster.
- Utilizing pypy, I am able to translate some of the more repetitive tasks into machine language.
- Code is run till the error norm is less than $10^{-5}$.
- We then sample values within an epsilon neighborhood of our point to determine if we have converged to a minimum.
Example of Output
Example of Output
A Continuum of Sheared Checkerboard Lattices
Future Work

- I will continue analyzing lattice configuration and their effects under various strains.
- Though not present in this talk a lot of my time was focused on trying to tackle this problem analytically. I will to continue this side of the work now that I have tools to test out and motivate theories.
Thank You

- Prof. Mao for an interesting project and always helpful mentorship
- My fellow REU students for a fun summer
- Myron, Jim and Angela for putting so much effort into organising a great program
- NSF for funding me and allowing me to have such an amazing experience