Buckling of Two Dimensional Maxwell Lattices

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Introduction

This summer I performed research with Prof. Xiaoming Mao in the field of Soft Matter Physics. One major goal of her work is to design novel mechanical metamaterials. Mechanical metamaterials are man-made structures with strange properties, such as tunable vibrational modes, negative Poisson ratio, negative thermal expansion that originate in the geometry of their unit cell. Often what guides such behavior is a floppy mode: a deformation that does not stretch or compress the links between constituent elements, that is a zero energy deformation.

In my research I have been tasked with investigating the out of plane buckling behavior of two dimensional maxwell lattices into 3-space. Maxwell lattices are a metastable collection of struts and links, meaning they are close to being rigid, but contain a limited amount of floppy modes. James Clerk Maxwell determined that sites linked by rigid struts cannot be stable unless the number of constraints, represented by the struts, exceeds the number of degrees of freedom of the sites minus the number of rigid translations and rotations.Thus the critical z, average incidence, separating unstable from stable frames is

$$z = 2d - \frac{d(d+1)}{N} \tag{1}$$

where d is the dimension of the embedding space and N is the number of vertices. This reduces to

$$z = 2d \tag{2}$$

in the limit of large N. It is in the regime of this critical value that maxwell lattices reside.

Though locally the interactions in our lattice can be described simply it is when we look at larger structures and see how each vertex interacts with the other that we see some emergent complexity in the floppy modes of a maxwell lattice. Finding exactly how to construct some kind of linkage that draws a prescribed curve (that is prescribing a floppy mode) is a difficult problem of great interest to physicists, mechanical engineers and algebraic geometers. In contrast to the analytic approach, numerical analysis of a maxwell lattice is only limited by available computational power. As such, a great deal of my time this summer was spent setting up and implementing a numerical scheme.

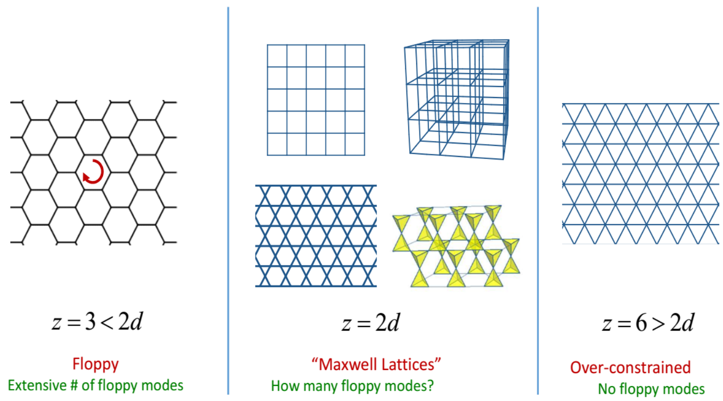


Figure 1: (a)hexagonal lattice (b) square, cubic, kagome, and pyrochlore lattice (c) triangular lattice

Determining Modes of Deformation

In order to determine the floppy modes in a maxwell lattices one may, to simplify matters, focus on the linear regime. As such, the standard method I will now present suffices only to determine the differential floppy modes. The linear response of a mechanical frame consisting of N point masses connected by N_b central force springs in d dimensions is fully described by the equilibrium rank 2 tensor $\mathbf{Q}$. This matrix relates the N_b dimensional vector of bond tensions t to the Nd dimensional vector of site forces f.

$$\mathbf{Q} \cdot t = f \tag{3}$$

$$f_i = t_{ij}\hat{n}_{ij} - t_{ki}\hat{n}_{ki} \tag{4}$$

where $\hat{n_{ij}}$ is the unit vector from i to j.

We now define the compatibility rank 2 tensor $\mathbf{C}$. This matrix relates the Nd dimensional vector of site displacements u to the N_b dimensional vector of bond extensions e

$$\mathbf{C} \cdot u = e \tag{5}$$

$$e_{ij} = \hat{n}_{ij}(u_i - u_j) \tag{6}$$

Elements in the matrix C are determined by the extension of the spring connecting sites i and j where, likewise, $\hat{n}_{ij}$ is the unit vector pointing from site i to site j.

We find that for these quantities the following is true,

$$rank(C) + N_o = Nd \tag{7}$$

$$rank(Q) + N_s = N_b \tag{8}$$

where N_o is the number of zero energy modes and N_s the number of states of self stress (equilibrium configurations). It can be shown $C^T = Q$. Thus utilizing the rank nullity theorem we may arrive at a more geometric determination of maxwell lattices given by the Maxwell-Calladine index theorem.

$$N_o - N_s = Nd - N_b \tag{9}$$

Code was written utilizing the above derivation to track the number of floppy modes. Utilizing this analysis I was also able to write code that numerically determines the floppy modes by following their differential counterparts, located in $ker(\mathbf{C})$, and ensuring that energy is nearly zero in each new configuration. This will be invaluable when studying even more complex lattices.

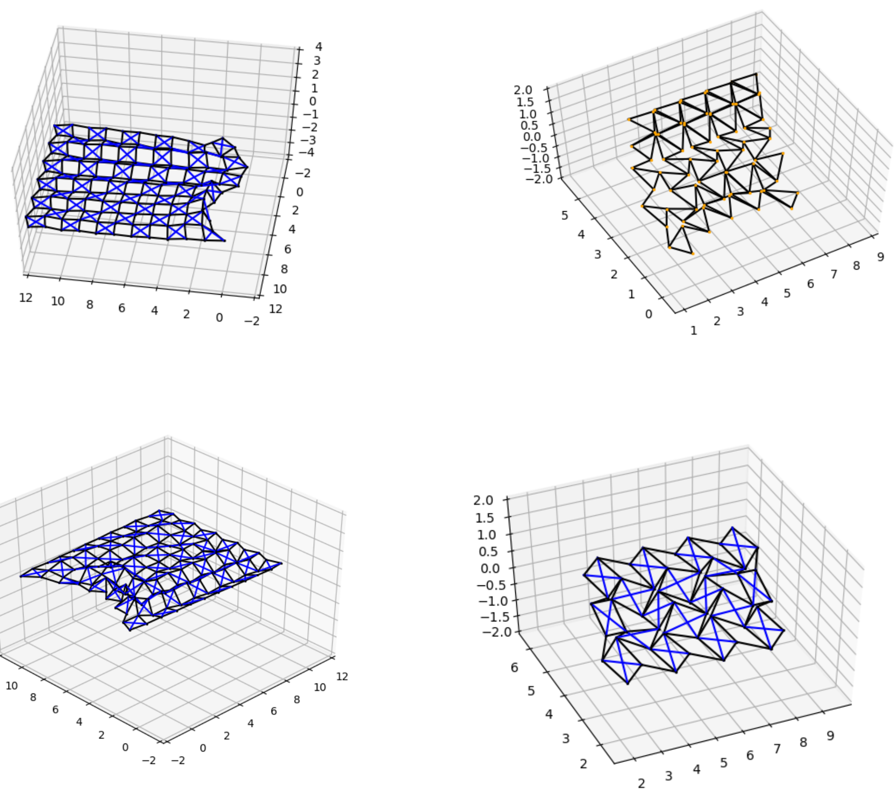
Minimal Energy Configurations

To begin, we really had very little idea how these out of plane deformations formed and how our floppy modes came into play to give us a minimal energy solution. Thus a major part of my work this summer was to create code that we could use to find the effects of various boundary conditions on certain maxwell lattices. This began with analysis of the checkerboard and kagome lattices and variations thereof. Both aforementioned lattices have a mode of uniform deformation and are auxetic lattices (i.e. their Poisson ratio is negative). We utilize the latter property with appropriate boundary conditions to create the buckling behavior.

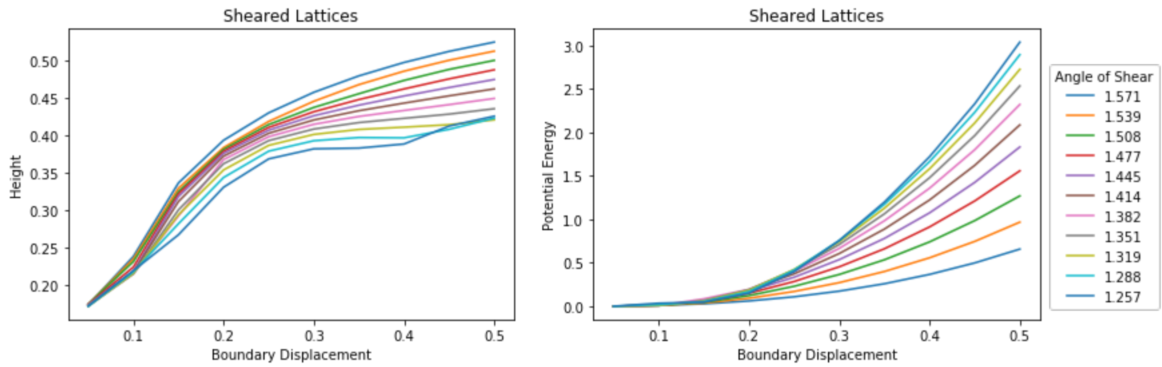
I began building off an example given to me that found the minimal energy solution in a computer algebra system. Run times for desired lattices would take multiple hours even for the smallest cases. A lot of this time was not just spent in minimizing our solution; defining the terms and simplifying the resulting equation to something usable itself was time intensive. I wrote code in python that would solve this problem numerically. My code is at its core a gradient descent algorithm. The code takes a point and utilizing a Lagrange polynomial (4 point central) finds the derivative.

To avoid the unnecessary terms, my code, given a lattice, generates the incidence matrix and uses that to remove the values that would cancel in calculating the derivative. To ensure convergence the step size is dynamically actuated. To further optimize the code I have implemented it in parallel so that now it runs on 24 cores in the cluster. Further altering my code so that I may run it with pypy, I was able to translate some of the more repetitive tasks into machine language instead of utilizing the python interpreter.

Code is run till the error norm is less than $10^{-}5$. We then sample values within an epsilon neighborhood of our point to determine if we have converged to a minimum. It would seem reasonable that we could achieve desired configurations utilizing appropriate the boundary deformations just as in the continuum picture. I ran this code to analyze boundary conditions on a plethora of different lattices. The following is example output of my code.



In the following I've plotted over 100 data points analyzing the continuum of sheared checkerboard lattices under a flat boundary displacement. Shearing the checkerboard lattice changes the principal vectors and so also alters the topological properties of our lattice giving us some interesting behavior.



Conclusion and Future Work

I have created a set of tools that allows one to quickly and accurately analyze the behavior of 2d maxwell lattices in 3-space. I've also used these tools to observe and analyze the buckling of a plethora of maxwell lattices. Though not present in this work a lot of my time was focused on trying to tackle this problem analytically. There is quite a bit of differential geometry and algebraic geometry involved to determine the zero energy paths, intersections of floppy modes, etc.. I will continue this analytic side of the work with the aid of these numerical tools to test out and motivate theories. This will all be done in hopes for forming some method to work backward from desired surfaces to underlying lattices and required strains.

References

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